The Statistical Energy-Frequency Spectrum of Random Disturbances

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A mathematical discussion of the statistical characteristics of Random Disturbances in terms of their "energy-frequency spectra" with applications to such typical disturbances as telegraph signals and "static".

IN a paper entitled "Selective Circuits and Static Interference" (*B. S. T. J.*, April, 1925) the writer discussed the "energy-frequency spectrum" (hereinafter precisely defined) of irregular random disturbances extending over a long interval of time. In view of our lack of even statistical information regarding static or atmospheric disturbances the specification of the energy-frequency spectrum, denoted by $R(\omega)$, was necessarily qualitative, and it was merely postulated that

"$R(\omega)$ is a continuous finite function of ω which converges to zero at infinity and is everywhere positive. It possesses no sharp maxima or minima and its variation with respect to $\omega(\omega = 2\pi f)$, where it exists, is relatively slow."

In a paper entitled "The Theory of the Schroteffekt,"[1] T. C. Fry deals with a similar problem, namely, the energy or "noise" absorbed in a vacuum tube from a stream of electrons with random time distribution. His method of attack is widely different from that of the present paper. In a more recent paper on "The Analysis of Irregular Motions with Applications to the Energy-Frequency Spectrum of Static and of Telegraph Signals" (Phil. Mag., Jan., 1929), G. W. Kenrick, by making certain hypotheses regarding the wave-form of the elementary disturbances whose aggregate is supposed to represent static interference, and by applying probability analysis, arrives at explicit formulas for the "statistical" or "expected" value of $R(\omega)$ for a number of different cases.

I

In the present paper the statistical or "expected" energy-frequency spectrum $R(\omega)$ of random disturbances is investigated by a method which is believed to be somewhat more general and direct than that of Kenrick.[2] The results are applicable to the Schroteffekt, telegraph

[1] *Jour. Franklin Inst.*, Feb., 1925.

[2] Kenrick's analysis is based on a formula derived originally by N. Wiener instead of proceeding directly from the Fourier integral.

signals and similar disturbances. The writer, however, concludes that their application to "static" or "atmospheric" disturbances is of questionable value owing partly to our lack of the necessary statistical information regarding such disturbances and also to the fact that they cannot be expected to have the "quasi-systematic" characteristics necessary to the application of probability theory.

The energy-frequency spectrum of a disturbance, as the concept is here employed, will now be defined. Let a disturbance $\Phi(t)$ exist in the epoch $0 \leqq t \leqq T$ and let

$$\begin{aligned} F(i\omega) &= C(\omega) + iS(\omega) \\ &= \int_0^T \Phi(t)e^{i\omega t}dt. \end{aligned} \tag{1}$$

Then, as shown in my paper referred to above,

$$\frac{1}{\pi}\int_0^\infty |F(i\omega)|^2 d\omega = \int_0^T \Phi^2 dt.$$

The energy-frequency spectrum is defined by the equation

$$G(\omega) = \operatorname*{Lim}_{T\to\infty} \frac{1}{\pi T} |F(i\omega)|^2, \tag{2}$$

so that

$$\int_0^\infty G(\omega)d\omega = \operatorname*{Lim}_{T\to\infty} \frac{1}{T}\int_0^T \Phi^2 dt. \tag{3}$$

It is on this last equation that the physical application of the concept of the energy-frequency spectrum rests; namely, that it determines the mean square value of $\Phi(t)$, as the epoch T is made indefinitely great. Its principal application in electrotechnics depends upon the further fact that, if $\Phi(t)$ represents an electromotive force applied to a network of impedance $Z(i\omega)$, the mean square current $\overline{I^2}$ absorbed by the network is given by [3]

$$\overline{I^2} = \operatorname*{Lim}_{T\to\infty} \frac{1}{T}\int_0^T I^2 dt = \int_0^\infty \frac{G(\omega)}{|Z(i\omega)|^2} d\omega. \tag{3a}$$

We now suppose that the function or disturbance $\Phi(t)$ is composed of a number N of elementary disturbances; thus

$$\Phi(t) = \sum_1^N a_m \phi_m(t - t_m), \tag{4}$$

[3] A somewhat more involved formula gives the mean power absorbed. See my paper referred to in the first paragraph.

the mth elementary disturbance being supposed zero until $t = t_m$. If we now write

$$f_m(i\omega) = c_m + is_m = \int_0^T \phi_m(t)e^{i\omega t}dt, \tag{5}$$

it is easy to show by the methods employed in my previous paper that

$$\begin{aligned} |F(i\omega)|^2 = \sum_1^N a_m^2 |f_m(i\omega)|^2 \\ + 2\sum_{m=1}^{N-1} \sum_{n=m+1}^{N} a_m a_n (c_m c_n + s_m s_n) \cos \omega(t_n - t_m) \\ + 2\sum_{m=1}^{N-1} \sum_{n=m+1}^{N} a_m a_n (c_m s_n - s_m c_n) \sin \omega(t_n - t_m). \end{aligned} \tag{6}$$

This is more compactly expressible as

$$\begin{aligned} |F(i\omega)|^2 = \sum_1^N a_m^2 |f_m(i\omega)|^2 \\ + 2\sum_{m=1}^{N-1} \sum_{n=m+1}^{N} \{a_m a_n \cdot f_m(i\omega) \cdot f_n(-i\omega)e^{i\omega(t_n - t_m)}\}_{\text{Real Part}}. \end{aligned} \tag{6a}$$

Now, obviously, if the amplitudes $a_1, \cdots, a_N$ and the wave form of the elementary functions $\phi_1, \cdots, \phi_N$ are specified, $G(\omega)$ is uniquely defined and determined by the preceding formula. This, however, is not the case in the problem under consideration, where at best the functions are specified only statistically by probability considerations. Under such circumstances, when the problem is correctly set and sufficient statistical information is furnished for its solution, we introduce the idea of the *statistical* energy-frequency spectrum $R(\omega)$ defined as follows:

The statistical energy-frequency spectrum $R(\omega)$ is equal to the weighted average of $G(\omega)$ for all possible values of $G(\omega)$, the weighting being in accordance with the probability of the occurrence of each particular possible value.

For example, the statistical value of a function $f(x_1, x_2, \cdots, x_n)$, where the variables $x_1, \cdots, x_n$ are defined only by probability considerations, is, in accordance with the foregoing definition,

$$\int_{-\infty}^{\infty} dx_1 p_1(x_1) \cdot \int_{-\infty}^{\infty} dx_2 p_2(x_2) \cdots \int_{-\infty}^{\infty} dx_n p_n(x_n) \cdot f(x_1, x_2, \cdots, x_n),$$

where $p_m(x_m)dx_m$ is the probability that x_m lies between x_m and $x_m + dx_m$.

To apply the foregoing concept and definition of the statistical value of a function to the problem at hand it is necessary to suppose that the typical impulse $f_m(i\omega)$ is a function of ω and certain parameters $\lambda_1, \lambda_2, \cdots, \lambda_n$, and that these parameters are statistically specified by probability considerations. Thus we suppose that $p_m(\lambda_m)d\lambda_m$ is the probability that λ_m lies between λ_m and $\lambda_m + d\lambda_m$. $G(\omega)$ will then be a function of ω and $\lambda_1, \lambda_2, \cdots, \lambda_n$, the amplitudes $a_1, \cdots, a_N$ being regarded as parameters, when defined by probability functions. We then have, in accordance with the foregoing,

$$R(\omega) = \int_{-\infty}^{\infty} d\lambda_1 p_1(\lambda) \cdot \int_{-\infty}^{\infty} d\lambda_2 p_2(\lambda_1) \times \cdots \int_{-\infty}^{\infty} d\lambda_n p_n(\lambda_n) G(\omega, \lambda_1, \lambda_2, \cdots, \lambda_n). \quad (7)$$

II

To apply the foregoing to the simplest possible case let us suppose that the elementary impulses are all identical; $a_1 = a_2 = \cdots a_N = 1$, and that their distribution in time is purely random. With these assumptions it follows at once from (6) that

$$R(\omega) = \frac{\nu}{\pi} |f(i\omega)|^2 + 2 \cdot \frac{\nu^2}{\pi} |f(i\omega)|^2 \frac{1 - \cos \omega T}{\omega^2 T}, \qquad T \to \infty. \quad (8)$$

If $f(i0) \neq 0$, this has a singularity at $\omega = 0$; however

$$\begin{aligned} \lim_{T \to \infty} \frac{1}{T} \int_0^T \Phi^2 dt &= \int_0^\infty R(\omega) d\omega \\ &= \nu \int \phi^2 dt + \nu^2 \left[\int \phi dt \right]^2. \end{aligned} \quad (9)$$

Here $\nu = N/T$ = mean frequency of occurrence of the elementary impulses. This formula is in entire agreement with Fry's results for the Schroteffekt (l.c.).

To consider a somewhat more involved problem, we shall suppose that the *durations* of the individual impulses and their *amplitudes* are distributed at random. We further denote the probability that the duration of any impulse, selected at random, lies between λ and $\lambda + d\lambda$ by $p(\lambda)d\lambda$. Correspondingly, $q(a)da$ denotes the probability that its amplitude lies between a and $a + da$. The durations and the amplitudes are then the statistically specified parameters.

We now postulate that $\Phi(t)$ is an *alternating series* of impulses of

the *same wave form;* i.e.

$$\Phi(t) = \sum_1^N (-1)^m a_m \phi_m(t - t_m),$$

$$\phi_m(t) = \phi(t), \qquad 0 \leq t \leq \lambda_m$$
$$= 0 \qquad t > \lambda_m,$$
$$t_m = \lambda_1 + \lambda_2 + \cdots + \lambda_{m-1},$$

and we denote the mean frequency of occurrence, N/T, by ν.

Substitution in the preceding formulas and straight-forward operations give

$$R(\omega) = \frac{\nu}{\pi} \int_0^\infty a^2 q(a) da \cdot \int_0^\infty |f(i\omega, \lambda)|^2 p(\lambda) d\lambda$$

plus the real part

$$\frac{2\nu}{\pi}\left[\int_0^\infty aq(a)da\right]^2 \cdot \int_0^\infty f(i\omega, \lambda)p(\lambda)e^{i\omega\lambda}d\lambda \cdot \int_0^\infty f(-i\omega, \lambda)p(\lambda)d\lambda$$
$$\times \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^N \sum_{n=m+1}^N (-1)^{n-m}\left[\int_0^\infty p(\lambda)e^{i\omega\lambda}d\lambda\right]^{n-m-1}. \tag{10}$$

If we write

$$\int_0^\infty p(\lambda)e^{i\omega\lambda}d\lambda = \rho(i\omega) = \rho, \tag{11}$$

we have by straightforward procedure

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^N \sum_{n=m+1}^N (-1)^{n-m}\left[\int_0^\infty p(\lambda)e^{i\omega\lambda}d\lambda\right]^{n-m-1} = \frac{-1}{1+\rho(i\omega)}, \tag{12}$$

whence

$$R(\omega) = \frac{\nu}{\pi} \int_0^\infty a^2 q(a) da \int_0^\infty |f(i\omega, \lambda)|^2 p(\lambda) d\lambda$$
$$- \frac{2\nu}{\pi}\left[\int_0^\infty aq(a)da\right]^2 \cdot \left\{\frac{U(\omega)\cdot V(\omega)}{1+\rho(i\omega)}\right\}_{\text{Real Part}}, \tag{13}$$

where

$$f(i\omega, \lambda) = \int_0^\lambda \phi(t)e^{i\omega t}dt = c(\omega, \lambda) + is(\omega, \lambda),$$

$$U(\omega) = \int_0^\infty f(i\omega, \lambda)p(\lambda)e^{i\omega\lambda}d\lambda, \tag{14}$$

$$V(\omega) = \int_0^\infty f(-i\omega, \lambda)p(\lambda)d\lambda.$$

If, on the other hand, we suppose that the impulses, instead of systematically alternating in sign, are equally likely to be positive or negative, the double summation term of (9) vanishes and

$$R(\omega) = \frac{\nu}{\pi} \int_{-\infty}^{\infty} a^2 q(a) da \cdot \int_0^{\infty} |f(i\omega, \lambda)|^2 p(\lambda) d\lambda. \tag{15}$$

This follows from the fact that the amplitude a is equally likely to be positive or negative. Consequently the integration with respect to da must be extended from $-\infty$ to $+\infty$ and, since by hypothesis $q(-a) = q(a)$, it follows that

$$\int_{-\infty}^{\infty} aq(a)da = 0.$$

To apply the preceding formulas to actual calculations, it is necessary to know the function $f(i\omega, \lambda)$ and in addition the probability functions involved. These latter may be supposed known from statistical data or calculable on theoretical assumptions. For example, if we assume that the times of incidence of the elementary disturbances are distributed entirely at random, the application of well-known probability theory gives $p(\lambda) = \nu e^{-\nu\lambda}$.

A third case is of interest. Here, instead of postulating that the termination of one impulse coincides with the start of the next (i.e. $t_{m+1} = t_m + \lambda_m$), we suppose that the times of incidence are entirely unrelated, and that the amplitudes are equally likely to be positive or negative. For this case the formula for $R(\omega)$ is formally identical with (15).

III

The foregoing analysis will now be applied to deriving what represents more or less accurately the statistical energy-frequency spectrum of telegraph signals. To this end we shall suppose that the elementary disturbance may have any one of three possible values (all equally probable), characterized by durations λ_1, λ_2, λ_3 and amplitudes a_1, a_2, a_3. The corresponding spectra of the elementary disturbances are then determined by the equations,

$$\begin{aligned} f_1(i\omega) &= \int_0^{\lambda_1} \phi(t) e^{i\omega t} dt, \\ f_2(i\omega) &= \int_0^{\lambda_2} \phi(t) e^{i\omega t} dt, \\ f_3(i\omega) &= \int_0^{\lambda_3} \phi(t) e^{i\omega t} dt. \end{aligned} \tag{16}$$

The application of the preceding analysis to this case gives

$$R(\omega) = \frac{\nu}{3\pi}\left(a_1^2|f_1(i\omega)|^2 + a_2^2|f_2(i\omega)|^2 + a_3^2|f_3(i\omega)|^2\right)$$

plus the *real part* of

$$\begin{aligned} \frac{2\nu}{9\pi}&\left(a_1 f_1(i\omega)e^{i\omega\lambda_1} + a_2 f_2(i\omega)e^{i\omega\lambda_2} + a_3 f_3(i\omega)e^{i\omega\lambda_3}\right) \\ &\times \left(a_1 f_1(-i\omega) + a_2 f_2(-i\omega) + a_3 f_3(-i\omega)\right) \\ &\times \lim_{N\to\infty} \frac{1}{N}\sum_{m=1}^{N-1}\sum_{n=m+1}^{N}\left[\tfrac{1}{3}\left(e^{i\omega\lambda_1} + e^{i\omega\lambda_2} + e^{i\omega\lambda_3}\right)\right]^{n-m-1}. \end{aligned} \tag{17}$$

It is to be understood that the *real part* of the second term is alone to be retained.

If we write

$$x = \tfrac{1}{3}\left(e^{i\omega\lambda_1} + e^{i\omega\lambda_2} + e^{i\omega\lambda_3}\right),$$

$$\frac{1}{N}\sum\sum\left[\tfrac{1}{3}\left(e^{i\omega\lambda_1} + e^{i\omega\lambda_2} + e^{i\omega\lambda_3}\right)\right]^{n-m-1} = \frac{1}{1-x}\left(\frac{N-1}{N} - \frac{x}{N}\frac{1-x^{N-1}}{1-x}\right)$$

and

$$\begin{aligned} \lim_{N\to\infty}\frac{1}{N}\sum\sum &= \frac{1}{1-x} & x < 0 \\ &= \frac{N}{2} & x = 1. \end{aligned}$$

There is therefore an infinity at $\omega = 0$, as we should expect. Its measure, however, is finite.

The preceding is merely an example which admits of extension to more complicated types of signals, as will be obvious to the reader. For example, the probabilities of the elementary signals need not be the same and their number need not be restricted to three.

IV

In all the cases discussed above it will be observed that the disturbance is "quasi-systematic" in the sense that the elementary disturbances are all of the same wave-form differing only in duration and amplitude. Indeed, some such assumptions as these are essential to the application of the mathematical theory. In the case of atmospheric disturbances we have no reason to suppose any such quasi-systematic character exists. Furthermore, even if for the sake of argument, we suppose that the elementary disturbances, which make up static, have a common wave form at the point at which they

originate, they would vary widely in this respect after arriving at a common receiver. The writer is therefore of the opinion that the quotation from his previous paper appearing at the start of this article, represents all that can safely be said regarding the spectrum of static and that our present knowledge is insufficient to justify the application of probability analysis to the problem. All that we can say is that the part of $R(\omega)$ which contributes to "static interference" is simply

$$\lim_{N \to \infty} \frac{\nu}{\pi} \cdot \frac{1}{N} \sum_{1}^{N} a_m^2 |f_m(i\omega)|^2,$$

a result deducible from (6) and in agreement with the conclusion of my original paper (l.c.). It is here supposed that the times of incidence are distributed at random. This formula, however, supplies no useful information in the absence of data regarding the wave forms and amplitudes of the individual disturbances.